

A RECURSIVE BOUND FOR A KAKEYA-TYPE MAXIMAL OPERATOR

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ABSTRACT. A (d, k) set is a subset of \mathbb{R}^d containing a translate of every k -dimensional plane. Bourgain showed that for $2^{k-1} + k \geq d$, every (d, k) set has positive Lebesgue measure. We give an L^p bound for the corresponding maximal operator.

1. INTRODUCTION

A measurable set $E \subset \mathbb{R}^d$ is said to be a (d, k) set if it contains a translate of every k -dimensional plane in \mathbb{R}^d . Once the definition is given, the question of the minimum size of a (d, k) set arises. This question has been extensively studied for the case $k = 1$, the Kakeya sets. It is known that there exist Kakeya sets of measure zero, and these are called Besicovitch sets. It is conjectured that all Besicovitch sets have Hausdorff dimension d . For $k \geq 2$, it is conjectured that (d, k) sets must have positive measure, i.e. that there are no (d, k) Besicovitch sets. These size estimates are related to L^p bounds on two maximal operators which we define below.

Let $G(d, k)$ denote the Grassmannian manifold of k -dimensional linear subspaces of \mathbb{R}^d . For $L \in G(d, k)$ we define

$$\mathcal{N}^k[f](L) = \sup_{x \in \mathbb{R}^d} \int_{x+L} f(y) dy$$

where we will only consider functions f supported in the unit ball $B(0, 1) \subset \mathbb{R}^d$.

A limiting and rescaling argument shows that if \mathcal{N}^k is bounded for some $p < \infty$ from $L^p(\mathbb{R}^d)$ to $L^1(G(d, k))$, then (d, k) sets must have positive measure. By testing \mathcal{N}^k on the characteristic function of $B(0, \delta)$, $\chi_{B(0, \delta)}$, one sees that such a bound may only hold for $p \geq \frac{d}{k}$. For L in $G(d, k)$ and $a \in \mathbb{R}^d$ define the δ plate centered at a , $L_\delta(a)$, to be the δ neighborhood in \mathbb{R}^d of the intersection of $B(a, \frac{1}{2})$ with $L + a$. Fixing L , considering $\mathcal{N}^k \chi_{L_\delta(0)}$, and using the fact that the dimension of $G(d, k)$ is $k(d - k)$ we see that a bound into $L^q(G(d, k))$ can only hold for $q \leq kp$. This leads to the following conjecture, where the case $k = 1$ is excluded due to the existence of Besicovitch sets.

Conjecture 1.1. For $2 \leq k < d, p > \frac{d}{k}, 1 \leq q \leq kp$

$$\|\mathcal{N}^k f\|_{L^q(G(d, k))} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

It is also useful to consider a generalization of the Kakeya maximal operator, defined for $L \in G(d, k)$ by

$$\mathcal{M}_\delta^k[f](L) = \sup_{a \in \mathbb{R}^d} \frac{1}{\mathcal{L}^d(L_\delta(a))} \int_{L_\delta(a)} f(y) dy$$

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where \mathcal{L}^d denotes Lebesgue measure on \mathbb{R}^d . Using an argument analogous to that in Lemma 2.15 of [2], one may see that a bound

$$(1) \quad \|\mathcal{M}_\delta^k f\|_{L^1(G(d,k))} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$

where $\alpha > 0$ and $p < \infty$, implies that the Hausdorff dimension of any (d, k) set is at least $d - \alpha$. Considering $\mathcal{M}_\delta^k \chi_{B(0,\delta)}$ and $\mathcal{M}_\delta^k \chi_{L_\delta(0)}$, we formulate

Conjecture 1.2. *For $k \geq 1, p < \frac{d}{k}, q \leq (d - k)p'$*

$$\|\mathcal{M}_\delta^k f\|_{L^q(G(d,k))} \lesssim \delta^{k - \frac{d}{p}} \|f\|_{L^p(\mathbb{R}^d)}.$$

In [4] Falconer showed that \mathcal{N}^k is bounded from $L^{\frac{d}{k} + \epsilon}(\mathbb{R}^d)$ to $L^1(G(d, k))$ for $k > \frac{d}{2}$. Later, in [2], Bourgain used a Keakeya maximal operator bound combined with an L^2 estimate of the x -ray transform to show that \mathcal{N}^k is bounded from $L^p(\mathbb{R}^d)$ to $L^p(G(d, k))$ for $(d, k, p) = (4, 2, 2 + \epsilon)$ and $(d, k, p) = (7, 3, 3 + \epsilon)$. He then showed, using a recursive metric entropy estimate, that for $d \leq 2^{k-1} + k$, \mathcal{N}^k is bounded for a large unspecified p . Substituting in the proof Katz and Tao's more recent bound for the Keakeya maximal operator in [9]

$$(2) \quad \|\mathcal{M}_\delta^1 f\|_{L^{n + \frac{3}{4}}(G(n, 1))} \lesssim \delta^{-\frac{3(n-1)}{4n+3}} \|f\|_{L^{\frac{4n+3}{7}}(\mathbb{R}^n)}$$

one now sees that this holds for $k > k_{cr}(d)$ where $k_{cr}(d)$ solves $d = \frac{7}{6}2^{k_{cr}-1} + k_{cr}$.

By Hölder's inequality, the following holds for any k -plate L_δ and positive f

$$\int_{L_\delta} f \, dx \lesssim \delta^{\frac{d-k}{r'}} \left(\int_{L^\perp} \left(\int_{L+y} f(x) \, d\mathcal{L}^k(x) \right)^r d\mathcal{L}^{d-k}(y) \right)^{\frac{1}{r}}.$$

Combining this with the $L^q(L^r)$ bounds on the k -plane transform proved by Christ in Theorem A of [3], we see that Conjecture 1.2 holds with $p \leq \frac{d+1}{k+1}$. Except for a factor of $\delta^{-\epsilon}$, the same bound for \mathcal{M}_δ^k was proven with $k = 2$ by Alvarez in [1] using a geometric-combinatorial “bush”-type argument. More recently, also see [7]. For dimension estimates of sets containing planes in directions corresponding to certain submanifolds of $G(4, 2)$, see [10].

Our main result is the following.

Theorem 1.1. *Suppose $4 \leq k < d$ and $k_{cr}(d) < k$. Then*

$$(3) \quad \|\mathcal{N}^k f\|_{L^p(G(d,k))} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

for f supported in the unit ball and $p > \frac{d-1}{2}$. If, additionally, we have $k - j > k_{cr}(d - j)$ for some integer j in $[1, k - 4]$, then we may take $p > \frac{d-3}{2+j} + 1$.

The number $p = \frac{d-3}{2+j} + 1$ is approximate, and may be slightly improved through careful numerology. We prove Theorem 1.1 by combining a recursive bound of \mathcal{M}_δ^k with Bourgain's L^2 estimate. This recursive bound is based on Bourgain's metric entropy argument, but is carried out in a manner which is more efficient for L^p estimates. For $k \leq k_{cr}(d)$ this method may be adapted to give the following bound on \mathcal{M}_δ^k

Theorem 1.2. *Suppose $2 \leq k \leq k_{cr}(d)$. Then*

$$\|\mathcal{M}_\delta^k f\|_{L^{\frac{d}{2}}(G(d,k))} \lesssim \delta^{-\frac{2}{d}(\frac{3(d-k)}{7(2^{k-2})} - 1 + \epsilon)} \|f\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}.$$

Finally, if $k + 1 < k_{cr}(d + 1)$ then it is preferable not to use the L^2 bound, giving

Theorem 1.3. *For $2 \leq k$*

$$(4) \quad \|\mathcal{M}_\delta^k f\|_{L^{d+1}(G(d,k))} \lesssim \delta^{-\frac{2}{d+1}(\frac{3(d-k)}{7(2^{k-1})} + \epsilon)} \|f\|_{L^{\frac{d+1}{2}}(\mathbb{R}^d)}.$$

From Theorems 1.2 and 1.3, we see that the Hausdorff dimension of any (d, k) set is at least

$$\min(d, d - \frac{3(d-k)}{7(2^{k-2})} + 1, d - \frac{3(d-k)}{7(2^{k-1})}).$$

It should be noted that the dimension estimate provided by only applying Theorem 1.3 is also a direct consequence of the metric entropy estimate in [2]. However, to the best of the author's knowledge, it has not previously appeared in the literature, even without the improvement permitted by (2).

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2. PRELIMINARIES

We start with the definition of the measure we will use on $G(d, k)$. Fix any $L \in G(d, k)$. For a Borel subset F of $G(d, k)$ let

$$\mathcal{G}^{(d,k)}(F) = \mathcal{O}(\{\theta \in O(d) : \theta(L) \in F\})$$

where \mathcal{O} is normalized Haar measure of the orthogonal group on \mathbb{R}^d , $O(d)$. Typically we will omit d and k , denoting the measure by \mathcal{G} . By the transitivity of the action of $O(d)$ on $G(d, k)$ and the invariance of \mathcal{O} , it is clear that the definition is independent of the choice of L . Also note that \mathcal{G} is invariant under the action of $O(d)$. By the uniqueness of uniformly-distributed measures (see [8], pages 44-53), \mathcal{G} is the unique normalized Radon measure on $G(d, k)$ invariant under $O(d)$.

It will be necessary to use two alternate formulations of \mathcal{G} . For each ξ in \mathbb{S}^{d-1} let $T_\xi : \xi^\perp \rightarrow \mathbb{R}^{d-1}$ be an orthogonal linear transformation. Then T_ξ^{-1} identifies $G(d-1, k-1)$ with the $k-1$ dimensional subspaces of ξ^\perp . Now, define $T : \mathbb{S}^{d-1} \times G(d-1, k-1) \rightarrow G(d, k)$ by

$$T(\xi, L) = \text{span}(\xi, T_\xi^{-1}(L)).$$

Choosing T_ξ continuously on the upper and lower hemispheres of \mathbb{S}^{d-1} , T^{-1} identifies the Borel subsets of $G(d, k)$ with the completion of the Borel subsets of $\mathbb{S}^{d-1} \times G(d-1, k-1)$. Under this identification, by uniqueness of rotation invariant measure, we have

$$(5) \quad \mathcal{G}(F) = \sigma^{d-1} \times \mathcal{G}^{(d-1, k-1)}(T^{-1}(F)).$$

where σ^{d-1} denotes normalized surface measure on the unit sphere.

It is also true that any invertible linear map $U : \mathbb{R}^d \rightarrow \mathbb{R}^d$ acts on $G(d, k)$. We will need to know how \mathcal{G} varies under this action. Again using the invariance of \mathcal{G} , we observe that

$$(6) \quad \mathcal{G}(F) = c\mathcal{L}^{kd}(\{(v_1, \dots, v_k) : v_j \in B(0, 1) \subset \mathbb{R}^d \text{ and } \text{span}(v_1, \dots, v_k) \in F\}).$$

Using (6) and noting that, for $0 \neq r \in \mathbb{R}$,

$$\text{span}(v_1, \dots, v_k) \in F \Leftrightarrow \text{span}(rv_1, \dots, rv_k) \in F,$$

we see that

$$\frac{|\det(U)|^k}{\|U\|^{kd}} \mathcal{G}(F) \leq \mathcal{G}(U(F)) \leq |\det(U)|^k \|U^{-1}\|^{kd} \mathcal{G}(F)$$

where $\|\cdot\|$ denotes the operator norm of a linear map. Since $|\det(U)| \leq \|U\|^d$ and $|\det(U)| = |\det(U^{-1})|^{-1}$, we have

$$(7) \quad (\|U\| \cdot \|U^{-1}\|)^{-kd} \mathcal{G}(F) \leq \mathcal{G}(U(F)) \leq (\|U\| \cdot \|U^{-1}\|)^{kd} \mathcal{G}(F).$$

Remark. One should know that there have been two incorrect proofs published on the subject of $(d, 2)$ sets. The first, in [5], is well known and it is of the claim that there are no Besicovitch $(d, 2)$ sets for any d . The second, in [6], is of the claim that $(d, 2)$ sets have Hausdorff dimension d for every d . Since it is quite recent, we will observe where the error is made. In the main construction, a 2-plate P^δ is isolated which intersects a large number of other 2-plates $\{P_k^\delta\}$. Then a $\frac{\delta}{\rho}$ separated set $\{e_i\} \subset (\mathbb{S}^{d-1} \cap P^\perp)$ is chosen, and the set of 3-plates $\{\Pi_i^{\tilde{C}^\delta}\}$ is considered where each $\Pi_i^{\tilde{C}^\delta}$ has the same center as P^δ and is in the direction $\text{span}(P, e_i)$. The aim is to show that each P_k^δ is contained in one of the $\Pi_i^{\tilde{C}^\delta}$. However, it is only shown that for each $y \in P_k^\delta$ there is an i so that $y \in \Pi_i^{\tilde{C}^\delta}$ and hence

$$P_k^\delta \subset \bigcup_i \Pi_i^{\tilde{C}^\delta}.$$

The only assumption placed on the P_k is that their distance from P is approximately ρ (where $\rho \gg \delta$). For $d \geq 4$ if we let $P = \text{span}(v_1, v_2)$ and $P_k = \text{span}(\sqrt{1 - \rho^2}v_1 + \rho v_3, \sqrt{1 - \rho^2}v_2 + \rho v_4)$ where the v_j are orthonormal, it can be seen that P_k satisfies this assumption. However P_k^δ cannot be contained in any such $\Pi_i^{\tilde{C}^\delta}$.

3. A RECURSIVE MAXIMAL OPERATOR BOUND

Our main argument is in the proof of Proposition 3.1 below.

Proposition 3.1. *Suppose $k \geq 2$, $2 \leq p \leq d+1$, $p \leq r \leq \frac{p(d-1)}{(p-2)}$, and $\frac{pr}{p+r} \leq q \leq r$. Then a bound for \mathcal{M}_δ^{k-1} on $L^p(\mathbb{R}^{d-1})$ of the form*

$$(8) \quad \|\mathcal{M}_\delta^{k-1} f\|_{L^q(G(d-1, k-1))} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^{d-1})}$$

implies the bound of \mathcal{M}_δ^k on $L^{\tilde{p}}(\mathbb{R}^d)$

$$\|\mathcal{M}_\delta^k f\|_{L^{\tilde{q}}(G(d, k))} \lesssim \delta^{-\frac{\tilde{\alpha}}{\tilde{p}}} \|f\|_{L^{\tilde{p}}(\mathbb{R}^d)}$$

with

$$\tilde{\alpha} = \frac{r\alpha}{p+r} + \epsilon, \quad \tilde{p} = \frac{p(d-1) + 2r}{p+r}, \quad \text{and} \quad \tilde{q} = \frac{\tilde{p}q(p+r)}{pr}.$$

For our applications we will always take $q = p$. It is then useful to note that the bound given by Proposition 3.1 is that which would result from interpolation between certain L^2 and L^{d-1} bounds, namely:

$$\frac{\tilde{\alpha}}{\tilde{p}} = \frac{\beta\alpha}{2} + \epsilon, \quad \frac{1}{\tilde{p}} = \frac{\beta}{2} + \frac{(1-\beta)}{d-1}, \quad \frac{1}{\tilde{q}} = \frac{\beta}{2} + \frac{(1-\beta)}{\infty}$$

where $\beta = \frac{2r}{p(d-1)+2r}$. For $\alpha < d - 2k$ this is better than the bound given by interpolation between the case $r = p$ and the known (sharp) L^2 bound. However, \tilde{p} still never seems to be optimal relative to $\tilde{\alpha}$ in the sense of Conjecture 1.2, which

may be explained by the fact that we expect \mathcal{M}_δ^k to be bounded independently of δ for $p > \frac{d}{k}$ rather than $p > d - 1$.

The choice $r = p$ yields the greatest reduction of α , giving $\tilde{\alpha} = \frac{1}{2}\alpha + \epsilon$. However, this also gives a relatively large $\tilde{p} = \frac{d+1}{2}$. Alternately, choosing $r = \frac{p(d-1)}{(p-2)}$ gives a small reduction of α and a relatively large reduction of p , with $\tilde{p} = \frac{p(d-1)}{d-2+p-1}$. Observe that with this choice of r and $m \leq d - 2$

$$(9) \quad p = \frac{(d-1)-1}{m} + 1 \text{ gives } \tilde{p} \leq \frac{d-1}{m+1} + 1.$$

Since interpolation with the L^∞ bound does not affect $\tilde{\alpha}$, we may take $\tilde{p} = \frac{d-1}{m+1} + 1$ for purposes of iteration.

To obtain Theorem 1.3, we use $k - 1$ applications of Proposition 3.1 with $r = p$. We start with the bound (2) for $n = (d - k + 1)$, except that we take $q_0 = p_0 = \frac{4n+3}{7}$ which is permitted by Hölder's inequality. This gives the value $\alpha_0 = \frac{3(n-1)}{7} + \epsilon = \frac{3(d-k)}{7} + \epsilon$. After one application of Proposition 3.1, we have $p_1 = \frac{(n+1)+1}{2}$, $q_1 = (n+1) + 1$, and $\alpha_1 = \frac{\alpha_0}{2} = \frac{3(d-k)}{2 \cdot 7} + \epsilon$. We use Hölder's inequality again, to take $q = p$, before another application of Proposition 3.1. Continuing this process gives (4).

To prove Proposition 3.1, we will need the following lemma which gives a sort of parameterization of disjoint pairs of separated elements of a subset of an interval.

Lemma 3.1. *Suppose $\Gamma \subset [-1, 1]$. Then for some integer l with $\mathcal{L}^1(\Gamma) \lesssim 2^{-l} \leq 1$ we have*

$$\mathcal{L}^1(S) \gtrsim \frac{1}{1 + |\log(\mathcal{L}^1(\Gamma))|} \mathcal{L}^1(\Gamma)$$

where

$$S = \{t \in [0, 2^{-l}) : |\Gamma \cap \{t + \mathbb{Z}2^{-l}\}| \geq 2\}.$$

Proof. For each integer $l \geq -1$ let

$$S_l = \{t \in [0, 2^{-l}) : \sum_{j=0}^{2^{l+1}-1} \chi_\Gamma(t - 1 + j2^{-l}) \geq 2\}$$

and

$$M_l = \int_{S_l} \sum_{j=0}^{2^{l+1}-1} \chi_\Gamma(t - 1 + j2^{-l}) dt.$$

Choose l_0 so that $\frac{\mathcal{L}^1(\Gamma)}{4} < 2^{-l_0} \leq \frac{\mathcal{L}^1(\Gamma)}{2}$. Note that

$$(10) \quad M_{l_0} \geq \frac{\mathcal{L}^1(\Gamma)}{2},$$

and that

$$(11) \quad M_{-1} = 0.$$

We want to observe that for some integer $l \in (-1, l_0]$,

$$(12) \quad \mathcal{L}^1(S_l) \geq \frac{\mathcal{L}^1(\Gamma)}{16l_0}.$$

To see this, note that for every l ,

$$(13) \quad M_l - M_{l-1} \leq 2\mathcal{L}^1(S_l).$$

Hence, if (12) does not hold for any $l \in (-1, l_0]$ then by (10), (13), and induction we have

$$M_l \geq \frac{\mathcal{L}^1(\Gamma)}{2} - (l_0 - l) \frac{\mathcal{L}^1(\Gamma)}{8l_0}$$

for every $l \in [-1, l_0]$. This is impossible by (11), proving (12). The lemma follows since $l_0 \lesssim 1 + |\log(\mathcal{L}^1(\Gamma))|$. \square

In the next lemma we show that the maximal operator \mathcal{M}_δ^k is local in the sense that we only need to prove bounds for functions supported in a ball.

Lemma 3.2. *For $q \geq p$ and $r > 0$ the bound $\|\mathcal{M}_\delta^k f\|_{L^q(G(d,k))} \leq C\|f\|_{L^p(\mathbb{R}^d)}$ for all f supported in $B(0, r)$ implies the bound $\|\mathcal{M}_\delta^k f\|_{L^q(G(d,k))} \leq \tilde{C}C\|f\|_{L^p(\mathbb{R}^d)}$ for all $f \in L^p(\mathbb{R}^d)$ where \tilde{C} is independent of δ .*

Proof. Assume f is positive. Note that since the bound holds for functions supported in the ball of radius r centered at 0, it holds for functions supported in any ball of radius r . Pick a covering $\{B(x_j, r)\}_{j=1}^\infty$ of \mathbb{R}^d where each point in \mathbb{R}^d is contained in only a finite number, say $c_{d,r}$, of the balls. Then any plate $L_\delta(a)$ touches at most $\tilde{c}_{d,r}$ of the balls. So for any L ,

$$\mathcal{M}_\delta^k f(L) \leq \tilde{c}_{d,r} \sup_j \mathcal{M}_\delta^k \chi_{B(x_j, r)} f(L) \leq \tilde{c}_{d,r} \left(\sum_j (\mathcal{M}_\delta^k \chi_{B(x_j, r)} f(L))^q \right)^{\frac{1}{q}}$$

Interchanging L^q and l^q and using our bound:

$$\begin{aligned} \|\mathcal{M}_\delta^k f(L)\|_{L^q(G(d,k))} &\leq \tilde{c}_{d,r} \left(\sum_j \|\mathcal{M}_\delta^k \chi_{B(x_j, r)} f\|_{L^q(G(d,k))}^q \right)^{\frac{1}{q}} \\ &\leq \tilde{c}_{d,r} C \left(\sum_j \|f\|_{L^p(B(x_j, r))}^q \right)^{\frac{1}{q}} \leq \tilde{c}_{d,r} C \left(\sum_j \|f\|_{L^p(B(x_j, r))}^p \right)^{\frac{1}{p}} \\ &\leq \tilde{c}_{d,r} c_{d,r}^{\frac{1}{p}} C \|f\|_{L^p(\mathbb{R}^d)} \end{aligned}$$

\square

Proof of Proposition 3.1. We will prove the restricted weak-type estimate for sets supported in $B(0, 1)$. This will give the full estimate for functions supported in $B(0, 1)$ by interpolation. The general case then follows by Lemma 3.2, since $\tilde{q} \geq \tilde{p}$.

We will only consider $\delta \leq \frac{1}{2}$. Let $E \subset B(0, 1) \subset \mathbb{R}^d$. Fix $0 < \lambda \leq 1$ and let

$$F = \{L \in G(d, k) : \mathcal{M}_\delta^k[\chi_E](L) > \lambda\}.$$

We need to show that

$$(14) \quad \mathcal{L}^d(E) \gtrsim \delta^{\frac{\alpha r}{p+r} + \epsilon} \lambda^{\tilde{p}} \mathcal{G}(F)^{\frac{\tilde{p}}{q}}.$$

By the trivial L^1 bound, there is a $c > 0$ so that (14) is satisfied for $\lambda \lesssim \delta^c$. Thus, we may assume that $|\log(\lambda)| \lesssim |\log(\delta)|$.

Instead of dealing directly with F , we will use its factorization via T^{-1} . Let

$$\tilde{F} = T^{-1}(F) \subset (\mathbb{S}^{d-1} \times G(d-1, k-1))$$

and $\tilde{\mathcal{G}} = \sigma^{d-1} \times \mathcal{G}^{(d-1, k-1)}$. Then, by (5), we have $\tilde{\mathcal{G}}(\tilde{F}) = \mathcal{G}(F)$.

Let $\{e_1, \dots, e_d\}$ be an orthonormal basis of \mathbb{R}^d . For each integer $i \in [1, d]$ let $W_i = \{\xi \in \mathbb{S}^{d-1} : |\langle \xi, e_i \rangle| \geq \frac{1}{\sqrt{d}}\}$. Then $\mathbb{S}^{d-1} = \bigcup_i W_i$, and thus for some i

$$(15) \quad \int_{W_i} \int_{G(d-1, k-1)} \chi_{\tilde{F}} dM d\xi \geq \frac{1}{d} \tilde{\mathcal{G}}(\tilde{F}) \gtrsim \mathcal{G}(F).$$

After renumbering assume that $i = d$.

Let $H = \text{span}(e_1, \dots, e_{d-1})$ and define, for $\xi \in W_d$, the projection along ξ onto H

$$P_\xi(x) = x - \frac{\langle x, e_d \rangle}{\langle \xi, e_d \rangle} \xi.$$

Henceforth, consider $G(d-1, k-1)$ as the set of $k-1$ -planes in the particular copy H of \mathbb{R}^{d-1} . We want to observe that if $L = \text{span}(\xi, M)$ where $\xi \in W_d$ and $M \in G(d-1, k-1)$, then for any $a \in \mathbb{R}^d$ we have

$$(16) \quad P_\xi(L_\delta(a)) \subset cM_\delta(P_\xi(a))$$

where $M_\delta(P_\xi(a)) \subset H$ is a $(k-1)$ -plate and c depends only on d . To see this we first note that any point $l \in L_\delta(a)$ can be written

$$l = a + b\xi + m + w,$$

where $b \in \mathbb{R}, m \in M, w \in L^\perp, |m| \lesssim \sqrt{d}$ and $|w| \leq \delta$. Then

$$P_\xi l = P_\xi a + m + P_\xi w.$$

But since $\text{dist}(\xi, H) \geq \frac{1}{\sqrt{d}}$ and $|w| \leq \delta$ it follows that

$$|P_\xi w| = \left| w - \frac{\langle w, e_d \rangle}{\langle \xi, e_d \rangle} \xi \right| \leq \delta(1 + \sqrt{d}) \approx: c\delta.$$

Thus $P_\xi(L_\delta(a))$ is contained in the $c\delta$ neighborhood, $cM_\delta(P_\xi(a))$, of $P_\xi(a) + (M \cap B(0, c))$.

For every $t \in \mathbb{R}$ let $H_t = H + te_d$ and $E_t = E \cap H_t$. Note that P_ξ is an isometry from H_t to H , giving

$$(17) \quad \mathcal{L}^{d-1}(E_t \cap L_\delta(a)) = \mathcal{L}^{d-1}(P_\xi(E_t \cap L_\delta(a)) \cap cM_\delta(P_\xi(a))).$$

The set \tilde{F} consists of pairs (ξ, M) such that $\text{span}(\xi, T_\xi^{-1}(M)) \in F$. However, considering (16), we should be interested in pairs (ξ, M) such that $\text{span}(\xi, M) \in F$. We obtain a set of such pairs by letting

$$\overline{F} = \{(\xi, P_\xi \circ T_\xi^{-1}(\tilde{M})) : \xi \in W_d \text{ and } (\xi, \tilde{M}) \in \tilde{F}\}.$$

We will use our change of coordinates to estimate $\tilde{\mathcal{G}}(\overline{F})$. Note that, by the orthogonality of ξ and ξ^\perp , for $x \in H$

$$|x| \leq |P_\xi \circ T_\xi^{-1}(x)| \leq (1 + \sqrt{d})|x|.$$

Then $\|(P_\xi \circ T_\xi^{-1})^{-1}\| \leq 1$ and $\|P_\xi \circ T_\xi^{-1}\| \leq (1 + \sqrt{d})$. Thus, by (7) and (15)

$$(18) \quad \begin{aligned} & \int_{W_d} \mathcal{G}^{(d-1, k-1)}(\{M : (\xi, M) \in \overline{F}\}) d\xi \\ & \gtrsim \int_{W_d} \mathcal{G}^{(d-1, k-1)}(\{M : (\xi, M) \in \tilde{F}\}) d\xi \\ & \gtrsim \mathcal{G}(F). \end{aligned}$$

For $\xi \in W_d$ and $s \neq t \in [-1, 1]$, define the subset of H

$$B_\xi^{s,t} = P_\xi(E_s) \cap P_\xi(E_t).$$

We will use the assumed maximal operator bound to estimate

$$\left(\int_{W_d} \mathcal{L}^{d-1}(B_\xi^{s,t})^{\frac{r}{p}} d\xi \right)^{\frac{p}{r}}.$$

This will provide us with an estimate of $(\mathcal{L}^{d-1}(E_s)\mathcal{L}^{d-1}(E_t))^{\frac{p+r}{2r}}$, effectively reducing the exponent of δ , as we will now explain. Consider E_s and E_t as subsets of H by orthogonal projection. Then

$$B_\xi^{s,t} = \left(E_s \cap \left(E_t + \frac{s-t}{\langle \xi, e_d \rangle} \text{proj}_H(\xi) \right) \right) - \frac{s}{\langle \xi, e_d \rangle} \text{proj}_H(\xi)$$

and so

$$\mathcal{L}^{d-1}(B_\xi^{s,t}) = \chi_{E_s} * \chi_{-E_t} \left(\frac{s-t}{\langle \xi, e_d \rangle} \text{proj}_H(\xi) \right)$$

where we use $*$ to denote convolution in \mathbb{R}^{d-1} . Since $\text{dist}(W_d, H) > \frac{1}{\sqrt{d}}$

$$d\sigma^{d-1}(\xi) \lesssim |s-t|^{-(d-1)} d\mathcal{L}^{d-1} \left(\frac{s-t}{\langle \xi, e_d \rangle} \text{proj}_H(\xi) \right).$$

Thus by changing variables, Young's inequality, and the fact that $r \geq p$

$$\begin{aligned} (19) \int_{W_d} \left(\mathcal{L}^{d-1}(B_\xi^{s,t}) \right)^{\frac{r}{p}} d\xi &\lesssim |s-t|^{\frac{-(d-1)p}{r}} \left(\int_{\mathbb{R}^{d-1}} (\chi_{E_s} * \chi_{-E_t}(x))^{\frac{r}{p}} dx \right)^{\frac{p}{r}} \\ &\lesssim |s-t|^{\frac{-(d-1)p}{r}} (\mathcal{L}^{d-1}(E_s)\mathcal{L}^{d-1}(E_t))^{\frac{p+r}{2r}}. \end{aligned}$$

We want to use our known maximal operator bound to estimate an average over s and t of the left hand side of (19). For each $x \in H, \xi \in W_d$ let

$$\Gamma_{\xi,x} = \{t : x \in P_\xi(E_t)\}.$$

Then, if $(\xi, M) \in \overline{F}$ we have $L := \text{span}(\xi, M) \in F$ and hence for some $a_L \in \mathbb{R}^d$,

$$\begin{aligned} \lambda \delta^{d-k} &\lesssim \mathcal{L}^d(L_\delta(a_L) \cap E) = \int_{-1}^1 \int_{cM_\delta(P_\xi(a_L))} \chi_{P_\xi(E_t \cap L_\delta(a_L))} dx dt \\ &\leq \int_{cM_\delta(P_\xi(a_L))} \int_{-1}^1 \chi_{P_\xi(E_t)} dt dx = \int_{cM_\delta(P_\xi(a_L))} \mathcal{L}^1(\Gamma_{\xi,x}) dx \end{aligned}$$

where the first equality follows from (17). Thus, considering $\mathcal{L}^1(\Gamma_{\xi,x})$ as a function of x ,

$$\mathcal{M}_\delta^{k-1}[\mathcal{L}^1(\Gamma_{\xi,c.})](M) \gtrsim \lambda.$$

Since (ξ, M) was an arbitrary element of \overline{F} and $r \geq q$ we now have by (18)

$$(20) \quad \left(\int_{W_d} \left(\int_{G(d-1,k-1)} (\mathcal{M}_\delta^{k-1}[\mathcal{L}^1(\Gamma_{\xi,c.})](M))^q dM \right)^{\frac{r}{q}} d\xi \right)^{\frac{1}{r}} \gtrsim \lambda \mathcal{G}(F)^{\frac{1}{q}}.$$

On the other hand, applying our assumed maximal operator bounds gives

$$(21) \quad \left(\int_{W_d} \left(\int_{G(d-1, k-1)} (\mathcal{M}_\delta^{k-1}[\mathcal{L}^1(\Gamma_{\xi, c.})](M))^q dM \right)^{\frac{r}{q}} d\xi \right)^{\frac{1}{r}} \\ \lesssim \delta^{-\frac{\alpha}{p}} \left(\int_{W_d} \left(\int_{\mathbb{R}^{d-1}} \mathcal{L}^1(\Gamma_{\xi, x})^p dx \right)^{\frac{r}{p}} d\xi \right)^{\frac{1}{r}}.$$

Let

$$Z = \{(\xi, x) \in W_d \times \mathbb{R}^{d-1} : \mathcal{L}^1(\Gamma_{\xi, x}) \gtrsim \frac{\lambda}{2}\}$$

and note that (20) and (21) still hold if we replace $\mathcal{L}^1(\Gamma_{\xi, x})$ by $\chi_Z \mathcal{L}^1(\Gamma_{\xi, x})$. For each $(\xi, x) \in Z$, we may apply Lemma 3.1 to $\Gamma_{\xi, x}$ obtaining an $l_{\xi, x}$ such that

$$\lambda \lesssim 2^{-l_{\xi, x}} \leq 1 \\ \text{and } \mathcal{L}^1(S_{\xi, x}^{l_{\xi, x}}) \gtrsim \frac{1}{1 + |\log(\lambda)|} \mathcal{L}^1(\Gamma_{\xi, x}) \\ \text{where } S_{\xi, x}^l = \{t \in [0, 2^{-l}) : |\{t + \mathbb{Z}2^{-l}\} \cap \Gamma_{\xi, x}| \geq 2\}.$$

Now,

$$\chi_Z \mathcal{L}^1(\Gamma_{\xi, x}) = \chi_Z \mathcal{L}^1(\Gamma_{\xi, x}) \sum_{i=1}^{C(1+|\log \lambda|)} \chi_{\{i\}}(l_{\xi, x})$$

and thus, combining (20) and (21) we may choose l_0 so that

$$\left(\int_{W_d} \left(\int_{\mathbb{R}^{d-1}} \chi_{\{l_0\}}(l_{\xi, x}) \chi_Z \mathcal{L}^1(\Gamma_{\xi, x})^p dx \right)^{\frac{r}{p}} d\xi \right)^{\frac{1}{r}} \gtrsim \frac{1}{1 + |\log(\lambda)|} \delta^{\frac{\alpha}{p}} \lambda \mathcal{G}(F)^{\frac{1}{q}}$$

and hence

$$(22) \quad \left(\int_{W_d} \left(\int_{\mathbb{R}^{d-1}} \mathcal{L}^1(S_{\xi, x}^{l_0})^p dx \right)^{\frac{r}{p}} d\xi \right)^{\frac{1}{r}} \gtrsim \frac{1}{(1 + |\log(\lambda)|)^2} \delta^{\frac{\alpha}{p}} \lambda \mathcal{G}(F)^{\frac{1}{q}}.$$

Recalling the appropriate definitions, we see that

$$S_{\xi, x}^{l_0} = \{t : x \in \bigcup_{i \neq j} B_\xi^{t+i2^{-l_0}, t+j2^{-l_0}}\}$$

where i and j range over $\mathbb{Z} \cap [-2^{l_0}, 2^{l_0})$. This gives

$$(23) \quad \mathcal{L}^1(S_{\xi, x}^{l_0}) = \int_0^{2^{-l_0}} \sup_{i \neq j} \chi_{B_\xi^{t+i2^{-l_0}, t+j2^{-l_0}}}(x) dt.$$

Noting that the $L_{\mathbb{S}^{d-1}}^r L_{\mathbb{R}^d}^p L_{\mathbb{Z}^2}^1 L_{\mathbb{Z}^2}^\infty$ norm is dominated by the $L_{\mathbb{R}}^1 L_{\mathbb{Z}^2}^p L_{\mathbb{S}^{d-1}}^r L_{\mathbb{R}^d}^p$ norm, we may combine (22) and (23), obtaining

$$\frac{1}{(1 + |\log(\lambda)|)^2} \delta^{\frac{\alpha}{p}} \lambda \mathcal{G}(F)^{\frac{1}{q}} \\ \lesssim \int_0^{2^{-l_0}} \left(\sum_{i \neq j} \left(\int_{W_d} \mathcal{L}^{d-1}(B_\xi^{t+i2^{-l_0}, t+j2^{-l_0}})^{\frac{r}{p}} d\xi \right)^{\frac{1}{r}} \right)^{\frac{p}{r}} dt.$$

Now, combining this with (19), we have

$$2^{-l_0 \frac{(d-1)}{r}} \frac{1}{(1 + |\log(\lambda)|)^2} \delta^{\frac{\alpha}{p}} \lambda \mathcal{G}(F)^{\frac{1}{q}} \\ \lesssim \int_0^{2^{-l_0}} \left(\sum_{i \neq j} (\mathcal{L}^{d-1}(E_{t+i2^{-l_0}}) \mathcal{L}^{d-1}(E_{t+j2^{-l_0}}))^{\frac{p+r}{2r}} \right)^{\frac{1}{p}} dt.$$

Finally,

$$\int_0^{2^{-l_0}} \left(\sum_{i \neq j} (\mathcal{L}^{d-1}(E_{t+i2^{-l_0}}) \mathcal{L}^{d-1}(E_{t+j2^{-l_0}}))^{\frac{p+r}{2r}} \right)^{\frac{1}{p}} dt \\ \leq \int_0^{2^{-l_0}} \left(\sum_i \mathcal{L}^{d-1}(E_{t+i2^{-l_0}})^{\frac{p+r}{2r}} \right)^{\frac{2}{p}} dt$$

and by Hölder's inequality and the conditions $2 \leq p \leq r$

$$\int_0^{2^{-l_0}} \left(\sum_i \mathcal{L}^{d-1}(E_{t+i2^{-l_0}})^{\frac{p+r}{2r}} \right)^{\frac{2}{p}} dt \leq 2^{-l_0(1-\frac{2}{p})} \mathcal{L}^d(E)^{\frac{p+r}{rp}}.$$

Summarizing

$$2^{-l_0(\frac{(d-1)}{r} + \frac{2}{p} - 1)} \frac{1}{(1 + |\log(\lambda)|)^2} \delta^{\frac{\alpha}{p}} \lambda \mathcal{G}(F)^{\frac{1}{q}} \lesssim \mathcal{L}^d(E)^{\frac{p+r}{rp}}.$$

Since $2^{-l_0} \gtrsim \lambda$ and $\frac{d-1}{r} + \frac{2}{p} - 1 \geq 0$ we have

$$\lambda^{\frac{p(d-1)+2r}{rp}} \delta^{\frac{\alpha}{p} + \epsilon} \mathcal{G}(F)^{\frac{1}{q}} \lesssim \mathcal{L}^d(E)^{\frac{p+r}{rp}}$$

or

$$\delta^{\frac{\alpha r}{p+r} + \epsilon} \lambda^{\frac{p(d-1)+2r}{p+r}} \mathcal{G}(F)^{\frac{pr}{q(p+r)}} \lesssim \mathcal{L}^d(E).$$

□

4. THE L^2 METHOD

Reducing α by a factor of two, as in Proposition 3.1, is not a substantial gain for small α . The following proposition gives $\tilde{\alpha} = \alpha - 1$ with $\alpha \geq 1$ and a bound for \mathcal{N}^k with $\alpha < 1$. It is proved using Bourgain's technique from Propositions 3.3 and 3.20 of [2] in which he showed bounds for \mathcal{N}^k with $(d, k) = (4, 2)$ and $(d, k) = (7, 3)$. For completeness we will repeat the argument.

Proposition 4.1. *Suppose $k, p \geq 2$ and that a bound for \mathcal{M}_δ^{k-1} on $L^p(\mathbb{R}^{d-1})$ of the form*

$$\|\mathcal{M}_\delta^{k-1} f\|_{L^p(G(d-1, k-1))} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^{d-1})}$$

is known. Then if $\alpha \geq 1$ we have the bound

$$(24) \quad \|\mathcal{M}_\delta^k f\|_{L^p(G(d, k))} \lesssim \delta^{-\frac{\alpha-1+\epsilon}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$

for $f \in L^p(\mathbb{R}^d)$. If $\alpha < 1$ we have the bound

$$(25) \quad \|\mathcal{N}^k f\|_{L^p(G(d, k))} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

for $f \in L^p(\mathbb{R}^d)$ supported in $B(0, 1)$.

To obtain Theorem 1.1, we start from an application of Theorem 1.3 with $k_0 = k - (2 + j)$ and $d_0 = d - (2 + j)$. After using Hölder's inequality on the left side, this gives

$$(26) \quad \|\mathcal{M}_\delta^{k_0} f\|_{L^{\frac{d_0+1}{2}}(G(d_0, k_0))} \lesssim \delta^{-\left(\frac{d_0+1}{2}\right)^{-1} \left(\frac{3(d-k)}{7(2^{k-(2+j)-1})} + \epsilon\right)} \|f\|_{L^{\frac{d_0+1}{2}}(\mathbb{R}^{d_0})}.$$

The condition $k - j > k_{cr}(d - j)$ ensures that

$$\frac{3(d-k)}{7 \cdot 2^{k-(2+j)-1}} + \epsilon < 2,$$

and hence further reduction in α is unnecessary. Thus, with our j “spare” iterations, we apply Proposition 3.1 with the maximum r to give a reduction in p . Noting that $\frac{d_0+1}{2}$ satisfies the left equation in (9) with $m = 2$, we start from (26) to obtain after the first iteration

$$\|\mathcal{M}_\delta^{k_1} f\|_{L^{\frac{d_1-1}{3}+1}(G(d_1, k_1))} \lesssim \delta^{-\left(\frac{d_1-1}{3}+1\right)^{-1} \left(\frac{3(d-k)}{7(2^{k-(2+j)-1})} + \epsilon\right)} \|f\|_{L^{\frac{d_1-1}{3}+1}(\mathbb{R}^{d_1})},$$

where $k_1 = k_0 + 1 = k - (2 + (j - 1))$, and $d_1 = d_0 + 1 = d - (2 + (j - 1))$. In fact, there is some additional improvement in α and p which we ignore. After $j - 1$ further iterations, we obtain

$$\|\mathcal{M}_\delta^{k_j} f\|_{L^{\frac{d_j-1}{j+2}+1}(G(d_j, k_j))} \lesssim \delta^{-\left(\frac{d_j-1}{j+2}+1\right)^{-1} \left(\frac{3(d-k)}{7(2^{k-(2+j)-1})} + \epsilon\right)} \|f\|_{L^{\frac{d_j-1}{j+2}+1}(\mathbb{R}^{d_j})},$$

where $k_j = k - 2$ and $d_j = d - 2$. We then apply Proposition 4.1 twice, using (24) the first time and (25) the second time, to obtain (3).

Theorem 1.2 is obtained by instead applying Theorem 1.3 with $k_0 = k - 1$ and $d_0 = d - 1$, and then applying (24) from Proposition 4.1 once.

To prove Proposition 4.1 we will need an $L^2(L^2)$ estimate for the x -ray transform which utilizes cancellation. For every $k > 0$ let ϕ^k be a positive Schwartz function on \mathbb{R}^k such that $\phi^k \geq 1$ on $B(0, \frac{3}{2})$ and the Fourier transform, $\hat{\phi}^k$, of ϕ^k has compact support. For $\xi \in \mathbb{S}^{d-1}$ and $x \in \xi^\perp$ define

$$\overline{f}_\xi(x) = \int \phi^1(t) f(x + t\xi) dt.$$

Lemma 4.1. *Suppose $\hat{f} \equiv 0$ in $B(0, R)$. Then*

$$\int_{\mathbb{S}^{d-1}} \int_{\xi^\perp} |\overline{f}_\xi(x)|^2 dx d\xi \lesssim \frac{1}{R} \|f\|_{L^2(\mathbb{R}^d)}^2.$$

Proof. Choose N so that $\hat{\phi}^1$ is supported in $(-N, N)$. Applying Plancherel's theorem to the partial Fourier transforms in the ξ and ξ^\perp directions, we have for every $\xi \in \mathbb{S}^{d-1}$

$$\int_{\xi^\perp} |\overline{f}_\xi(x)|^2 dx = \int_{\xi^\perp} \left| \int_{\mathbb{R}} \hat{\phi}^1(t) \hat{f}(\zeta + t\xi) dt \right|^2 d\zeta.$$

Considering the support of $\hat{\phi}^1$ and using Hölder's inequality we have

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \int_{\xi^\perp} \left| \int_{\mathbb{R}} \hat{\phi}^1(t) \hat{f}(\zeta + t\xi) dt \right|^2 d\zeta d\xi \\ & \leq 2N \|\hat{\phi}^1\|_{L^\infty}^2 \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} |\hat{f}(y)|^2 |\chi_{[-N,N]}(\langle y, \xi \rangle)|^2 dy d\xi. \end{aligned}$$

Then for any y

$$(27) \quad \int_{\mathbb{S}^{d-1}} \chi_{[-N,N]}(\langle y, \xi \rangle) d\xi = \sigma^{d-1} \left(\left\{ \xi : \text{dist}(\xi, y^\perp) \leq \frac{N}{|y|} \right\} \right) \lesssim \frac{N}{|y|}.$$

Since $|y| \geq R$ in the support of \hat{f} , we are done. \square

We will want to take advantage of the fact that the averaging operator \mathcal{M}_δ^k should tend to localize the Fourier transform. To this effect, we will define a modified version of our maximal operator. For $L \in G(d, k)$ let

$$\pi_\delta^L(x) = \phi^k(\text{proj}_L(x)) \delta^{-(d-k)} \phi^{(d-k)} \left(\text{proj}_{L^\perp} \left(\frac{x}{\delta} \right) \right).$$

Now, define

$$\widetilde{\mathcal{M}}_\delta^k[f](L) = \sup_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} \pi_\delta^L(a+x) f(x) dx.$$

Immediately, we see that for all positive f , $\mathcal{M}_\delta^k[f] \lesssim \widetilde{\mathcal{M}}_\delta^k[f]$. We will see that the reverse inequality also holds.

Let φ be a Schwartz function on \mathbb{R}^d so that $\widehat{\varphi} \equiv 1$ on $B(0, 1)$ and $\widehat{\varphi}$ is supported in $B(0, 2)$. For every $R > 0$ let $\varphi_R = R^d \varphi(R \cdot)$.

Lemma 4.2. *Suppose \hat{f} is supported in $B(0, R)$. Then for any k -plane $L \in G(d, k)$ and $a \in \mathbb{R}^d$ we have*

$$(28) \quad \int_{a+(L \cap B(0, \frac{1}{2}))} |f(x)| dx \lesssim \mathcal{M}_{\frac{1}{R}}^k[|f|](L)$$

while for any $\delta > 0$ there is the estimate

$$(29) \quad \mathcal{M}_\delta^k[|f|](L) \lesssim \mathcal{M}_{\frac{1}{R}}^k[|f|](L).$$

Also, without any assumptions on the support of \hat{f} ,

$$(30) \quad \widetilde{\mathcal{M}}_\delta^k[|f|](L) \lesssim \mathcal{M}_\delta^k[|f|](L).$$

Proof. The statement (29) follows from (28) by averaging. Inequality (30) can be proved by the same method used in the proof of (28). So we will only prove (28).

By our assumption on f , $f = f * \varphi_R$ so

$$\begin{aligned} \int_{a+(L \cap B(0, \frac{1}{2}))} |f(x)| dx &= \int_{a+(L \cap B(0, \frac{1}{2}))} |f * \varphi_R(x)| dx \\ &\leq \int_{\mathbb{R}^d} |\varphi_R(y)| \int_{a-y+(L \cap B(0, \frac{1}{2}))} |f(x)| dx dy. \end{aligned}$$

Let e_1, \dots, e_d be an orthonormal basis of \mathbb{R}^d where $L = \text{span}(\{e_1, \dots, e_k\})$. For each $z \in \mathbb{Z}^d$ let $b_z^R = (\frac{2}{\sqrt{dR}} z_1 e_1, \dots, \frac{2}{\sqrt{dR}} z_d e_d)$. Let $Q_1^R = L \cap B(0, \frac{1}{R})$ and $Q_2^R = L^\perp \cap B(0, \frac{1}{R})$.

Then

$$\begin{aligned}
& \int_{\mathbb{R}^d} |\varphi_R(y)| \int_{a-y+(L \cap B(0, \frac{1}{2}))} |f(x)| dx dy \\
& \leq \sum_{z \in \mathbb{Z}^d} \int_{b_z^R + Q_1^R \times Q_2^R} |\varphi_R(y)| \int_{a-y+(L \cap B(0, \frac{1}{2}))} |f(x)| dx dy \\
& \lesssim \sum_{z \in \mathbb{Z}^d} \sup_{y \in b_z^R + Q_1^R \times Q_2^R} |\varphi_R(y)| \int_{b_z^R + Q_1^R} \frac{1}{R^{d-k}} \mathcal{M}_{\frac{1}{R}}^k[|f|](L) dy' \\
& \lesssim \mathcal{M}_{\frac{1}{R}}^k[|f|](L) \sum_{z \in \mathbb{Z}^d} \frac{1}{R^d} \sup_{y \in b_z^R + Q_1^R \times Q_2^R} |\varphi_R(y)|.
\end{aligned}$$

But

$$(31) \quad \sum_{z \in \mathbb{Z}^d} \frac{1}{R^d} \sup_{y \in b_z^R + Q_1^R \times Q_2^R} |\varphi_R(y)| = \sum_{z \in \mathbb{Z}^d} \sup_{y \in b_z^1 + Q_1^1 \times Q_2^1} |\varphi_1(y)|$$

and the right-hand side of (31) is controlled independently of L and R since φ_1 is a Schwartz function. \square

Proof of Proposition 4.1. We will start by proving (24). It suffices to consider the case when f is positive and bounded. By Lemma 3.2 we may also assume that f is supported in $B(0, 1)$. Also we will only consider, say, $\delta \leq \frac{1}{2}$.

By (5), we need to show that

$$\left(\int_{\mathbb{S}^{d-1}} \int_{G(d-1, k-1)} \mathcal{M}_{\delta}^k[f](\text{span}(\xi, T_{\xi}^{-1}M))^p dM d\xi \right)^{\frac{1}{p}} \lesssim \delta^{-(\frac{\alpha-1}{p} + \epsilon)} \|f\|_{L^p(\mathbb{R}^d)}.$$

Note that, by our assumption on the support of f ,

$$(32) \quad \mathcal{M}_{\delta}^k[f](\text{span}(\xi, T_{\xi}^{-1}M)) \leq \mathcal{M}_{\delta}^{k-1}[\bar{f}_{\xi} \circ T_{\xi}^{-1}](M).$$

By a change of variables, the fact that T_{ξ}^{-1} is orthogonal, and Plancherel's theorem in one dimension,

$$(33) \quad \widehat{\bar{f}_{\xi} \circ T_{\xi}^{-1}}(\zeta) = \int_{\mathbb{R}} \widehat{\phi^1}(t) \hat{f}(T_{\xi}^{-1}\zeta + t\xi) dt.$$

Let $g = f * \varphi_{\frac{\delta}{8}}$. Then by (33), the support of $\widehat{\phi^1}$, and our restriction on δ ,

$$(34) \quad \widehat{\bar{g}_{\xi} \circ T_{\xi}^{-1}} \equiv \widehat{\bar{f}_{\xi} \circ T_{\xi}^{-1}} \text{ on } B(0, \frac{\tilde{c}}{\delta}).$$

Hence, using (34) for the equality and Lemma 4.2 for the last inequality

$$(35) \quad \begin{aligned} |\mathcal{M}_{\delta}^{k-1}[\bar{f}_{\xi} \circ T_{\xi}^{-1}](M)| & \lesssim |\widetilde{\mathcal{M}}_{\delta}^{k-1}[\bar{f}_{\xi} \circ T_{\xi}^{-1}](M)| = |\widetilde{\mathcal{M}}_{\delta}^{k-1}[\bar{g}_{\xi} \circ T_{\xi}^{-1}](M)| \\ & \lesssim \mathcal{M}_{\delta}^{k-1}[|\bar{g}_{\xi}| \circ T_{\xi}^{-1}](M). \end{aligned}$$

We will use the Littlewood-Paley decomposition of g . Let $\psi_0 = \varphi$ and for $j > 0$ let $\psi_j = 2^{jd} \varphi(2^j \cdot) - 2^{(j-1)d} \varphi(2^{j-1} \cdot)$. Note that

$$\sum_{j=1}^{\infty} \widehat{\psi_j} \equiv 1$$

and that for $j > 0$, $\widehat{\psi_j}$ is supported in the annulus centered at 0 with radii 2^{k-1} and 2^{k+1} . For each $j \geq 0$, let $g_j = g * \psi_j$. Then, considering the support of \hat{g}

$$(36) \quad g = \sum_{j=0}^{\log \frac{c}{\delta}} g_j.$$

Now, by (35) and (36)

$$(37) \quad \left(\int_{\mathbb{S}^{d-1}} \int_{G(d-1, k-1)} |\mathcal{M}_\delta^{k-1}[\bar{f}_\xi \circ T_\xi^{-1}](M)|^p dM d\xi \right)^{\frac{1}{p}} \\ \leq \sum_{j=0}^{\log \frac{c}{\delta}} \left(\int_{\mathbb{S}^{d-1}} \int_{G(d-1, k-1)} \mathcal{M}_\delta^{k-1}[|\bar{g}_{j\xi}| \circ T_\xi^{-1}](M)^p dM d\xi \right)^{\frac{1}{p}}.$$

Because each $\widehat{\bar{g}_{j\xi} \circ T_\xi^{-1}}$ is supported in $B(0, 2^{j+1})$, inequality (29) from Lemma 4.2 allows us to apply our assumed bound with $\delta \approx 2^{-j}$ to give

$$(38) \quad \int_{G(d-1, k-1)} \mathcal{M}_\delta^{k-1}[|\bar{g}_{j\xi}| \circ T_\xi^{-1}](M)^p dM \\ \lesssim \int_{G(d-1, k-1)} \mathcal{M}_{2^{-j}}^{k-1}[|\bar{g}_{j\xi}| \circ T_\xi^{-1}](M)^p dM \\ \lesssim 2^{j\alpha} \int_{\mathbb{R}^{d-1}} |\bar{g}_{j\xi} \circ T_\xi^{-1}|^p dx \lesssim 2^{j\alpha} \|g_j\|_{L^\infty}^{p-2} \int_{\xi^\perp} |\bar{g}_{j\xi}|^2 dx$$

where, for the last inequality, we use the assumption that $p \geq 2$. Because each \hat{g}_j is identically zero on $B(0, 2^{j-1})$, integrating (38) and using Lemma 4.1 gives

$$\int_{\mathbb{S}^{d-1}} \int_{G(d-1, k-1)} \mathcal{M}_\delta^{k-1}[|\bar{g}_{j\xi}| \circ T_\xi^{-1}](M)^p dM d\xi \lesssim 2^{j(\alpha-1)} \|g_j\|_{L^\infty}^{p-2} \|g_j\|_{L^2}^2.$$

Thus

$$(39) \quad \sum_{j=0}^{\log \frac{c}{\delta}} \left(\int_{\mathbb{S}^{d-1}} \int_{G(d-1, k-1)} \mathcal{M}_\delta^{k-1}[|\bar{g}_{j\xi}| \circ T_\xi^{-1}](M)^p dM d\xi \right)^{\frac{1}{p}} \\ \lesssim \sum_{j=0}^{\log \frac{c}{\delta}} 2^{\frac{j}{p}(\alpha-1)} \|g_j\|_{L^\infty}^{1-\frac{2}{p}} \|g_j\|_{L^2}^{\frac{2}{p}} \\ \lesssim \|f\|_{L^\infty}^{1-\frac{2}{p}} \|f\|_{L^2}^{\frac{2}{p}} \sum_{j=0}^{\log \frac{c}{\delta}} \left(2^{\frac{\alpha-1}{p}} \right)^j \lesssim \|f\|_{L^\infty}^{1-\frac{2}{p}} \|f\|_{L^2}^{\frac{2}{p}} \delta^{-\frac{\alpha-1}{p}}$$

Combining (32) (37) and (39), we see that it only remains to show

$$\|f\|_{L^\infty}^{1-\frac{2}{p}} \|f\|_{L^2}^{\frac{2}{p}} \lesssim \|f\|_{L_p}.$$

This will hold under the additional assumption that f is a characteristic function. Sacrificing an ϵ in the exponent, this is sufficient by interpolation.

The proof of (25) is identical except that we use (28) instead of (29), and in (36) we must sum to ∞ instead of $\log(\frac{c}{\delta})$. This will converge in the end, by our assumption $\alpha < 1$. \square

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